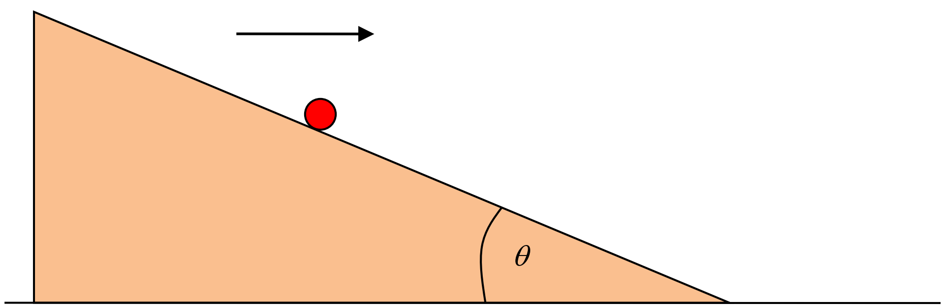
Teacher notes
Topic A

An instructive problem with Newton's second law

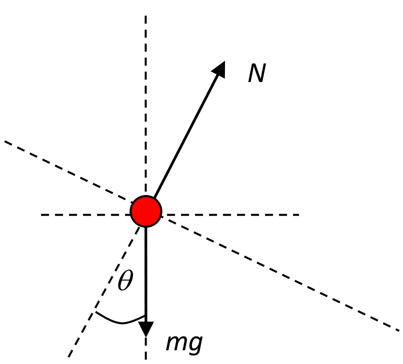
A wedge making an angle θ with the horizontal accelerates with constant acceleration a along a horizontal straight road.



A ball on the wedge remains at rest **relative to the wedge**. The wedge is frictionless.

Find the acceleration of the wedge for this to happen and discuss what happens when the wedge moves with a greater acceleration.

We begin by drawing the forces on the ball.



We deduce that

$$N\sin\theta = ma$$
$$N\cos\theta = mg$$

Hence, dividing side by side we get the result:

$$a = g\tan\theta$$

This means that $N\sin\theta = mg\tan\theta \Rightarrow N = \dfrac{mg}{\cos\theta} = mg\sec\theta$.

What happens to the ball if the wedge moves with an acceleration that is greater than $g\tan\theta$?

Now let $a = g\tan\theta + \alpha$ so that the acceleration is now greater than $g\tan\theta$ by α.

The equations

$$N\sin\theta = ma$$
$$N\cos\theta = mg$$

cannot both hold because they would lead to an acceleration $g\tan\theta$. It must be that the ball moves on the plane, so its acceleration is not the same as that of the wedge. Let the ball move up the plane with an acceleration β. The acceleration of the wedge *along the plane* is $(g\tan\theta + \alpha)\cos\theta$. The net acceleration of the ball along the plane relative to the ground is then $(g\tan\theta + \alpha)\cos\theta - \beta$. The only force with a component along the plane is the weight and so:

$$mg\sin\theta = m\big((\alpha + g\tan\theta)\cos\theta - \beta\big)$$
$$mg\sin\theta = m\alpha\cos\theta + mg\sin\theta - m\beta$$

Hence $\beta = \alpha\cos\theta$.

This says that the ball moves up the plane with acceleration $\alpha\cos\theta$.

The equation of motion is (we are looking along the horizontal direction and so the component of acceleration of the ball horizontally is $\beta\cos\theta$)

$$\begin{aligned}
N\sin\theta &= ma \\
&= m(\alpha + g\tan\theta - \beta\cos\theta) \\
&= m(\alpha + g\tan\theta - \alpha\cos^2\theta) \\
&= m(\alpha\sin^2\theta + g\tan\theta) \\
N &= m(\alpha\sin\theta + g\sec\theta)
\end{aligned}$$

As expected, the normal force increased as the acceleration of the wedge increased. As mentioned above both equations

$$N\sin\theta = ma$$
$$N\cos\theta = mg$$

cannot hold. Indeed, the second gives:

$$\begin{aligned} N\cos\theta - mg &= m(\alpha\sin\theta + g\sec\theta)\cos\theta - mg \\ &= m\alpha\sin\theta\cos\theta + mg - mg \\ &= m\alpha\sin\theta\cos\theta \end{aligned}$$

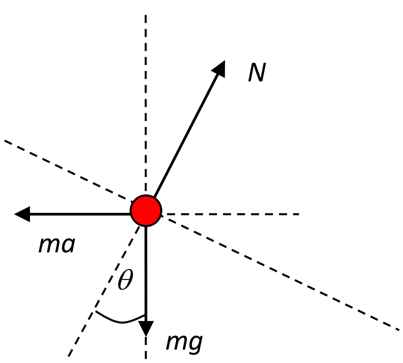
This means that there is vertical component of acceleration equal to $\alpha\sin\theta\cos\theta$. This is perfectly consistent with we found above: the ball has a component of acceleration $\alpha\cos\theta$ along the plane. This acceleration then has a component $(\alpha\cos\theta)\sin\theta$ in the vertical direction. It is all consistent.

You can sometimes see this in practice when driving a car in rainy weather. Raindrops on the windshield can be seen moving up the windshield when the car accelerates. The analogy with the problem we discussed here is not perfect, however. The raindrop problem is more complicated because there are surface tension forces to consider etc.

Solution using a non-inertial frame (Not for students!)

Using non-inertial frames and fictitious forces usually simplifies complicated problems. This is the case here.

The problem is best answered by making use of a non-inertial frame of reference, namely the frame in which the wedge is at rest. The price we pay for using such a frame is that we must include a fictitious force *ma* acting opposite to the acceleration of the wedge (when viewed in an inertial frame). In other words, the forces on the ball are now:



Before going on, let us see that we can get the previous answer for $a = g\tan\theta$ in the non-inertial frame as well.

In the non-inertial frame, the ball is in equilibrium and so

$$N\sin\theta = ma$$
$$N\cos\theta = mg$$

These are the same equations as before with the same result: $a = g\tan\theta$!

Now let $a = g\tan\theta + \alpha$ so that the acceleration is now greater than $g\tan\theta$ by α.

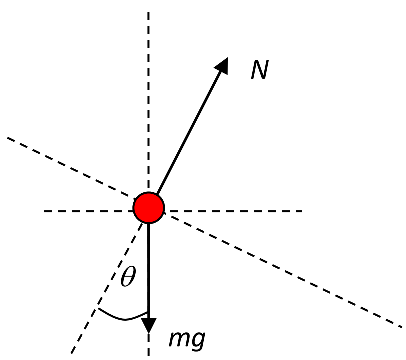
The components along the plane give a net force up the plane given by:

$$\begin{aligned} ma\cos\theta - mg\sin\theta &= m(g\tan\theta + \alpha)\cos\theta - mg\sin\theta \\ &= mg\tan\theta\cos\theta + m\alpha\cos\theta - mg\sin\theta \\ &= mg\sin\theta + m\alpha\cos\theta - mg\sin\theta \\ &= m\alpha\cos\theta \end{aligned}$$

In other words, *the ball will accelerate up the plane* with an acceleration given by $\alpha\cos\theta$.

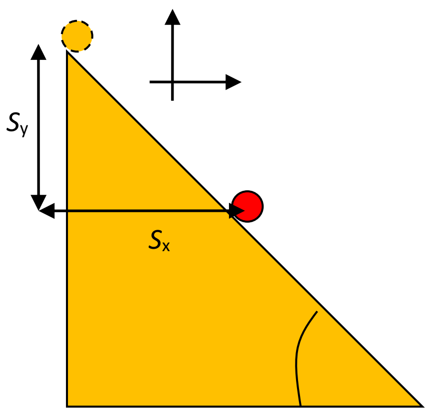
What follows is a problem that might be discussed with advanced students. But if makes a very nice connection with what we did at the beginning of this note.

We can now complicate this a bit more by asking: the wedge moves with acceleration *A*. What is the acceleration of the ball? To avoid cumbersome trigonometry, take $\theta = 45^\circ$.



$$ma_x = N\sin 45^\circ = \frac{N}{\sqrt{2}}$$
$$ma_y = N\cos 45^\circ - mg = \frac{N}{\sqrt{2}} - mg$$

The ball is in contact with the wedge all the time so we must have:



$S_x = \frac{1}{2}(a_x - A)t^2$ and $S_y = -\frac{1}{2}a_y t^2$ and since $\frac{S_y}{S_x} = \tan\theta = 1$ we have $a_y = A - a_x$

From the first equation $N = ma_x\sqrt{2}$ and substituting in the second:

$$ma_y = \frac{ma_x\sqrt{2}}{\sqrt{2}} - mg = ma_x - mg$$

Now use $a_y = A - a_x$ to find

$$m(A - a_x) = ma_x - mg$$
$$-a_x - a_x = -A - g$$
$$a_x = \frac{A+g}{2}$$

Hence

$$a_y = -(\frac{A+g}{2} - A) = \frac{A-g}{2}$$

Notice that if $A = g\tan\theta = g\tan 45^\circ = g$, then $a_y = 0$ and $a_x = g$. This is complete agreement with what we did at the beginning. If $A = g\tan 45^\circ = g$ the ball stays at rest **relative to the wedge**. From the analysis here we see that $a_y = 0$ so the height of the ball stays the same and $a_x = g$ which means the ball moves along with the wedge.